CAPILLARY WAVES IN A CHANNEL WITH A POLYGONAL BOTTOM

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Capillary waves of small amplitude on the surface of fluids with a constant depth have been well studied [1]. Below is proposed an approximate method for investigating capillary waves originating from the motion of fluid in a channel with a bottom which has a polygonal form.

1. Let us consider the steady potential flow of a perfect incompressible weightless fluid with a polygonal rigid boundary $ABCD$ and free surface KL in the plane $z = x + iy$ (Fig.1). The angles of slope of sections AB, BC and CD with respect to the x-axis are given. The fluid flows from A to D and at infinity upstream the velocity of its undisturbed motion is V_0 ; H is the depth of the flow.

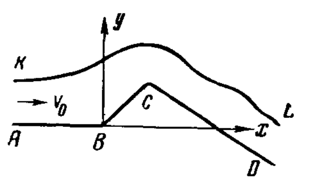

Fig. 1

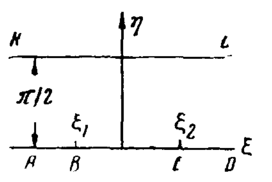

Fig. 2

Let R be the radius of curvature of the free surface (it is assumed positive if directed from the surface), T the coefficient of surface tension, ρ the density of the fluid, p the pressure in the fluid and p_0 the pressure in the atmosphere.

It is known that on the free surface

$$\frac{T}{R} = p_0 - p, \quad \text{or} \quad TV\frac{d\theta}{d\varphi} = \frac{\rho}{2}(V^2 - V_0^2) \quad (1.1)$$

Here V is the modulus of the velocity, and θ is the slope angle of the velocity with respect to the x-axis. Let $w = \varphi + i\psi$ be the complex flow potential.

A zone of breadth $\psi_0 = V_0 H$ corresponds to the region of flow in the w plane.

We reflect the w region upon the $\zeta = \xi + i\eta$ region with the aid of the linear transformation

$$\zeta = \frac{\pi}{2V_0H}\, w \tag{1.2}$$

In the ζ plane we obtain a zone of breadth $\frac{1}{2}\pi$; the free surface corresponds to the straight line $\eta = \frac{1}{2}\pi$ and the rigid boundary of the flow to the straight line $\eta = 0$

The solution of the problem can be represented in the form

$$z = \frac{2H}{\pi}\int \exp\,[f(\zeta)]\, d\zeta \qquad \left(f = \ln\frac{V_0\, dz}{dw} = r + i\theta\right) \tag{1.3}$$

Here $f(\zeta)$ is the Zhukovskii function.

We will regard the image of points B, C in the ζ plane as given, then on the straight line $\eta = 0$ the imaginary part of the function $f(\zeta)$ is known. On the straight line $\eta = \frac{1}{2}\pi$ the condition (1.1) should be fulfilled, it can be easily reduced to the form

$$\alpha\frac{\pi}{2}\frac{d\theta}{d\xi} = -\sinh r \qquad \left(\alpha = \frac{T}{\rho V_0^2 H}\right) \tag{1.4}$$

The dimensionless parameter α characterizes the degree of the influence of the capillarity. With $\alpha = 0$ the capillary forces are absent and on the free surface $r = 0$. For the case of small values of α (this case will be of interest later), on the free surface the value of r is small and we can approximate equality (1.4) by letting $\sinh r = r$. The error of such a substitution for small r has the order $^1/_6 r^3$.

Thus, we arrive at the following problem. It is required to determine the function $f(\zeta)$ which is analytical in the region $-\infty \leqslant \xi \leqslant \infty$, $0 \leqslant \eta \leqslant {}^1/_2\pi$, if on the straight line $\eta = 0$ the values of its imaginary part $\theta_0(\xi)$ are known, and on the straight line $\eta = \frac{1}{2}\pi$ the real and imaginary parts of the function are connected by the relation

$$r_1 = -\alpha\,\frac{\pi}{2}\,\frac{d\theta_1}{d\xi} \tag{1.5}$$

(by subscript o we will denote the functions r and θ for $\eta = 0$, and by subscript 1 for $\eta = \frac{1}{2}\pi$).

2. Writing down the expression for the function $f(\zeta)$ in terms of $\theta_0(\xi)$, $r_1(\xi)$, and subsequently $\theta_0(\xi)$, $\theta_1(\xi)$ [2], and subtracting from the first expression the second one, we can obtain Formula

$$2\int_{-\infty}^{\infty}\theta_0\frac{d\xi'}{1+e^{\xi'-\xi}} = 2\int_{-\infty}^{\infty}\theta_1\frac{d\xi'}{1+e^{2(\xi'-\xi)}} - \int_{-\infty}^{\infty}\frac{r_1}{\cosh(\xi'-\xi)}\,d\xi' \tag{2.1}$$

(detailed deduction of this formula is given in [3]). Substituting in (2.1) for r_1 its expression in terms of θ_1 from (1.5), we arrive at the integro-differential equation for θ_1

$$2\int_{-\infty}^{\infty}\theta_0\,\frac{d\xi'}{1+e^{\xi'-\xi}} = 2\int_{-\infty}^{\infty}\theta_1\,\frac{d\xi'}{1+e^{2(\xi'-\xi)}} + \alpha\,\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{d\theta_1}{d\xi}\operatorname{sech}(\xi'-\xi)\,d\xi' \tag{2.2}$$

Let $\zeta = \xi_1$, ξ_2 be the image of the points B and C in the ζ plane, and θ_{00}, θ_{01}, θ_{02} are the values of θ_0 for sections AB, BC and CD, respectively.

Applying to (2.2) the two-sided transform of Laplace [4], we obtain

$$\theta_1(\xi) = \frac{1}{2\pi i}\sum_{k=1}^{2}(\theta_{0k}-\theta_{0k-1})\,J_k, \qquad J_k = \int_{c-i\infty}^{c+i\infty} Q_k(p)\,dp \tag{2.3}$$

$$Q_k(p) = \frac{\exp\,[p(\xi-\xi_k)]}{p\,({}^1/_2\,\alpha\pi p\sin{}^1/_2\pi p + \cos{}^1/_2\pi p)}$$

Here the integration is carried out in the plane of auxiliary complex variable $p = \sigma + i\tau$ along any straight line, parallel to the imaginary axis and lying in the zone $0 < \sigma < 1$.

Using Jordan's lemma, we can prove that for $\xi - \xi_k > 0$ the integral of J_k is equal to the sum of the residues of the integrand at all poles located to the left of the line of integration, and for $\xi - \xi_k < 0$ the integral of J_k is equal to the sum of the residues taken with the inverse sign at all poles to the right of the line of integration.

The integrand $Q_k(p)$ has no poles outside of the coordinate axes $\sigma = 0$, $\tau = 0$. The point $p = 0$ will be a pole of first order; for $\tau = 0$, $\sigma \neq 0$ the function $Q_k(p)$ has poles of first order at the points $\sigma_n > 0$ and $\sigma_{-n} < 0$, and for $\sigma = 0$, $\tau \neq 0$ the function has the same poles at the points $+ i\tau_0$ and $- i\tau_0$ which are determined, respectively, from the conditions

$$-\alpha\pi\sigma_n / 2 = \cot(\pi\sigma_n / 2), \qquad \sigma_n = -\sigma_{-n}, \qquad 2n - 1 \leqslant \sigma_n \leqslant 2n$$

$$\alpha\pi\tau_0 / 2 = \coth(\pi\tau_0 / 2) \tag{2.4}$$

Taking into account (2.4), we can write down the residues of the function $Q_k(p)$ in the following form:

$$\operatorname*{res}_{p=\sigma_n} Q_k(p) = -\frac{2}{\pi}\frac{\exp[\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{\sigma_n[1-\alpha\sin^2(\pi\sigma_n/2)]}$$

$$\operatorname*{res}_{p=-\sigma_n} Q_k(p) = -\frac{2}{\pi}\frac{\exp[-\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{\sigma_n[1-\alpha\sin^2(\pi\sigma_n/2)]}$$

$$\operatorname*{res}_{p=i\tau_0} Q_k(p) = -\frac{2}{\pi}\frac{\exp[i\tau_0(\xi-\xi_k)]\sinh(\pi\tau_0/2)}{\tau_0[1+\alpha\sinh^2(\pi\tau_0/2)]},$$

$$\operatorname*{res}_{p=-i\tau_0} Q_k(p) = -\frac{2}{\pi}\frac{\exp[-i\tau_0(\xi-\xi_k)]\sinh(\pi\tau_0/2)}{\tau_0[1+\alpha\sinh^2(\pi\tau_0/2)]}$$

$$\operatorname*{res}_{p=0} Q_k(p) = 1$$

Thus, we obtain, that for $\xi - \xi_k \leqslant 0$

$$J_k = J_k^- = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp[\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{\sigma_n[1-\alpha\sin^2(\pi\sigma_n/2)]} \tag{2.5}$$

$$\frac{dJ_k^-}{d\xi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp[\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{1-\alpha\sin^2(\pi\sigma_n/2)} \tag{2.6}$$

and for $\xi - \xi_k \geqslant 0$

$$J_k = J_k^+ = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp[-\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{\sigma_n[1-\alpha\sin^2(\pi\sigma_n/2)]} - \frac{4}{\pi}\frac{\cos[\tau_0(\xi-\xi_k)]\sinh(\pi\tau_0/2)}{\tau_0[1+\alpha\sinh^2(\pi\tau_0/2)]} \tag{2.7}$$

$$\frac{dJ_k^+}{d\xi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp[-\sigma_n(\xi-\xi_k)]\sin(\pi\sigma_n/2)}{1-\alpha\sin^2(\pi\sigma_n/2)} + \frac{4}{\pi}\frac{\sin[\tau_0(\xi-\xi_k)]\sinh(\pi\tau_0/2)}{1+\alpha\sinh^2(\pi\tau_0/2)} \tag{2.8}$$

From Fig.3 it can be seen that for n increasing from 1 to infinity the quantity $\sin(\pi\sigma_n/2)$ will alternate its sign and monotonously decrease in modulus.

Therefore the series, entering Expressions (2.5) to (2.8), reprsent the alternate in sign terms monotonously decreasing in their absolute value.

In addition

$$\left.\frac{dJ_k^-}{d\xi}\right|_{\xi=\xi_k} = \left.\frac{dJ_k^+}{d\xi}\right|_{\xi=\xi_k} \qquad J_k^-|_{\xi=\xi_k} = J_k^+|_{\xi=\xi_k}$$

the second equality follows from the consideration of the residues of the function $Q_k(p)$ for $\xi = \xi_k$.

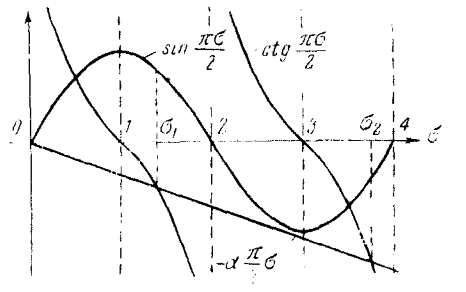

Fig. 3

From (2.3) we obtain Expressions for $\theta_1(\xi)$

$$\theta_1 = (\theta_{01} - \theta_{00}) J_1^- + (\theta_{02} - \theta_{01}) J_2^- \quad \text{for} \quad -\infty \leqslant \xi \leqslant \xi_1$$

$$\theta_1 = (\theta_{01} - \theta_{00}) J_1^+ + (\theta_{02} - \theta_{01}) J_2^- \quad \text{for} \quad \xi_1 \leqslant \xi \leqslant \xi_2$$

$$\theta_1 = (\theta_{01} - \theta_{00}) J_1^+ + (\theta_{02} - \theta_{01}) J_2^+ \quad \text{for} \quad \xi_2 \leqslant \xi \leqslant \infty$$

Analogous expressions are obtained from (1.5) for $r_1(\xi)$

$$r_1 = \alpha \frac{\pi}{2}\left[(\theta_{00} - \theta_{01}) \frac{dJ_1^-}{d\xi} + (\theta_{01} - \theta_{02}) \frac{dJ_2^-}{d\xi}\right] \quad \text{for} \quad -\infty \leqslant \xi \leqslant \xi_1$$

$$r_1 = \alpha \frac{\pi}{2}\left[(\theta_{00} - \theta_{01}) \frac{dJ_1^+}{d\xi} + (\theta_{01} - \theta_{02}) \frac{dJ_2^-}{d\xi}\right] \quad \text{for} \quad \xi_1 \leqslant \xi \leqslant \xi_2$$

$$r_1 = \alpha \frac{\pi}{2}\left[(\theta_{00} - \theta_{01}) \frac{dJ_1^+}{d\xi} + (\theta_{01} - \theta_{02}) \frac{dJ_2^+}{d\xi}\right] \quad \text{for} \quad \xi_2 \leqslant \xi \leqslant \infty$$

It is obvious that $r_1 \to 0$ for $\alpha \to 0$, and consequently the obtained formulas are applicable for the small values of α . If in (2.5),(2.7) we take $\alpha = 0$, and also taking into account that for this case $\sigma_n = 2n - 1$, we will have

$$J_k^- = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \exp[(2n-1)(\xi - \xi_k)] = \frac{2}{\pi} \tan^{-1}[\exp(\xi - \xi_k)]$$

$$J_k^+ = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \exp[(1-2n)(\xi - \xi_k)] = \frac{2}{\pi} \tan^{-1}[\exp(\xi - \xi_k)]$$

Hence

$$\theta_1 = \frac{2}{\pi} \sum_{k=1}^{2} (\theta_{0k} - \theta_{0k-1}) \tan^{-1}[\exp(\xi - \xi_k)] \qquad (-\infty \leqslant \xi \leqslant \infty)$$

It is easy to see that this formula gives the solution of the problem for the condition $r_1 = 0$.

3. Let us consider a fluid flow over a vertical step, i.e. assuming the following values: $\theta_{00} = \theta_{02} = 0$, $\theta_{01} = -{}^1/_2\,\pi$, $\xi_1 = -\beta$, $\xi_2 = \beta$. Then we will have in particular

$$r_1 = \alpha \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\exp[-\sigma_n(\xi + \beta)] - \exp[-\sigma_n(\xi - \beta)]}{1 - \alpha \sin^2(\pi\sigma_n/2)} \sin(\pi\sigma_n/2) + \frac{2\alpha\pi \sinh(\pi\tau_0/2)}{1 + \alpha \sinh^2(\pi\tau_0/2)} \sin\beta\tau_0 \cos\xi\tau_0 \qquad \text{for} \quad \beta \leqslant \xi \leqslant \infty \tag{3.1}$$

For large values of ξ (far from the step downstream)

$$r_1 = \frac{2\alpha\pi \sinh(\pi\tau_0/2)}{1 + \alpha \sinh^2(\pi\tau_0/2)} \sin\beta\tau_0 \cos\xi\tau_0, \qquad \theta_1 = -\frac{4\sinh(\pi\tau_0/2)}{\tau_0[1 + \alpha\sinh^2(\pi\tau_0/2)]} \sin\beta\tau_0 \sin\xi\tau_0$$

For $0 \leqslant \alpha \leqslant 0.25$ we can assume $\frac{1}{2}\alpha\pi\tau_0 = \coth(\pi\tau_0/2) = 1$ with great accuracy.
Therefore,

$$\frac{2\alpha\pi \sinh(\pi\tau_0/2)}{1+\alpha \sinh^2(\pi\tau_0/2)} = \frac{4\sinh(\pi\tau_0/2)}{\tau_0\,[1+\alpha\sinh^2(\pi\tau_0/2)]} = \frac{2\alpha\pi\sinh(1/\alpha)}{1+\alpha\sinh^2(1/\alpha)} \equiv k(\alpha)$$

and Formulas (3.1) reduce to the form

$$r_1 = k(\alpha)\sin\frac{2\beta}{\pi\alpha}\cos\frac{2\xi}{\pi\alpha}, \qquad \theta_1 = -k(\alpha)\sin\frac{2\beta}{\pi\alpha}\sin\frac{2\xi}{\pi\alpha}$$

We present the value of k for certain values of α.

$\alpha =$	$^1/_4$	$^1/_5$	$^1/_6$	$^1/_7$	$^1/_8$	$^1/_9$	0
$k =$	0.22909	0.08460	0.03114	0.01146	0.00422	0.00155	0

From (1.3) we have the parametric equations for the free surface away from the step

$$x = \alpha H\int \exp(h\cos u)\cos(h\sin u)\,du \qquad \left(h = k(\alpha)\sin\frac{2\beta}{\pi\alpha}, \quad u = \xi\tau_0\right)$$

$$y = -\alpha H\int \exp(h\cos u)\sin(h\sin u)\,du$$

Taking into account that h is small, it is convenient to compute the obtained integral, expanding the integrand in a power series of h. With accuracy to the order of h^3 we will have

$$x = \alpha H(u + h\sin u + {}^1/_4 h^2\sin 2u), \qquad y = \alpha H(h\cos u + {}^1/_2 h^2\cos 2u)$$

Hence it follows that the wave length λ and amplitude δ are

$$\lambda = 2\pi\alpha H, \qquad \delta = H\alpha k(\alpha)\left|\sin\frac{2\beta}{\pi\alpha}\right|$$

Thus, the wavelength does not depend on the height of the step, and the quantity δ for the monotonous change of the step height oscillates in the interval $0 \leqslant \delta \leqslant H\alpha\, k(\alpha)$ (the relation between the step height and the quantity β can be established approximately by solving the problem for the condition $r_1 = 0$). In exactly the same manner we can investigate the flow in corner, by placing $\xi_1 = \xi_2$. For this case we obtain

$$\lambda = 2\pi\alpha H, \qquad \delta = H\frac{|\theta_{00} - \theta_{01}|}{\pi}\alpha k(\alpha)$$

In general for any flow of the type investigated above the expression for wavelength λ remains invariant.

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